AN EXPANSION OF A SIMPLE ALTERNANT OF THE THIRD ORDER

THESIS

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THESIS

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I. Introduction

An alternating function is a function that changes only in sign when two of its variables are interchanged. An example is,

$$G(x,y) = x^2 - y^2;$$

for $$G(y,x) = y^2 - x^2 = -G(x,y).$$

The determinant

$$\begin{vmatrix} f_1(x) & f_1(y) & f_1(z) \\ f_2(x) & f_2(y) & f_2(z) \\ f_3(x) & f_3(y) & f_3(z) \end{vmatrix}$$

is an alternating function, since any two variables may be interchanged by interchanging the two columns containing these variables, thus changing the sign of the determinant. From this comes the definition that any determinant which is an alternating function is an alternant.

If each element of the alternant is a positive integral power of a variable, the determinant is called a simple alternant. A short way of indicating a simple alternant is to write the exponents in descending order. For example,

$$\begin{vmatrix} a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \\ a^5 & b^5 & c^5 & d^5 \\ a^6 & b^6 & c^6 & d^6 \end{vmatrix} = |6, 5, 3, 1|.$$

A symmetric function is a function in which any two of the variables may be interchanged without altering the function. An example of this is

$$F(a,b,c) = a^3 + b^3 + c^3 + a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + abc,$$

which may be written as

$$F(a,b,c) = \Sigma a^3 + \Sigma a^2b + \Sigma abc.$$

The object of this paper is to consider the expansion of $|r,\ q,\ p|$. This alternant can be arranged so that $r > q > p$ and the factor $(abc)^p$ divided out, leaving $|n,\ m,\ o|$ as the general simple alternant of the third order.

The first case to be considered is that of the difference-product, sometimes referred to as the Vandermonde determinant. This is $|2,\ 1,\ 0|$, which is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c-b).$$

The difference product must appear as a factor of the expansion of any simple alternant, because when any two of the variables are equated the value of the determinant reduces to zero. Likewise the expansion of this determinant must reduce to zero under the same operation. This is possible if and only if the expansion contains the factors $(b-a)$, $(c-a)$ and $(c-b)$. In this paper D will be used for $(b-a)(c-a)(c-b)$ and D' for $(b-a)(c-a)$.

From the foregoing discussion $|m,\ n,\ o| = D[g(a,b,c)]$. To determine the character of the factor $g(a,b,c)$, it is noticed that on interchanging any two variables, both $|m,\ n,\ o|$ and D change only in sign, leaving g(a,b,c) unaltered. This means that g(a,b,c) is a symmetric function.

II. Previous Work on Simple Alternants of the Third Order

In 1885 W. W. Johnson published a procedure for expanding a simple alternant of the third order.[1] His device was to divide the alternant by its difference-product and obtain a symmetric function plus the quotient of a simple alternant of lower degree and the difference-product. For example,

$$\frac{|5,3,0|}{|2,1,0|} = \sum a^3b^2 + \sum a^3bc + \sum a^2b^2c + abc\frac{|3,2,0|}{|2,1,0|}$$

$$= \sum a^3b^2 + \sum a^3bc + \sum a^2b^2c + abc\left[\sum ab + abc\frac{|1,1,0|}{|2,1,0|}\right]$$

$$= \sum a^3b^2 + \sum a^3bc + 2\sum a^2b^2c.$$

In 1886 Johnson published a method of determining the coefficient of each term of the symmetric function by referring to a hexagon.[2] During this same year Sir Thomas Muir presented another process, one involving combinations, for determining the coefficients obtained in Johnson's expansion.[3]

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[1] Wm. Woolsey Johnson, "On a Formula of Reduction for Alternants of the Third Order", American Journal of Mathematics, VII, 347-352 (Baltimore, 1885).

[2] Wm. Woolsey Johnson, "On a Geometrical Representation of Alternants of the Third Order and of Their Quotients When Divided by A(0,1,2)", Quarterly Journal of Pure and Applied Mathematics, XXI, 217-224 (London, 1886).

[3] Sir Thomas Muir, "On the Quotient of a Simple Alternant by the Difference Product of the Variables", Proceedings of the Royal Society of Edinburg, XIV, 433-435 (Edinburg, 1887).

III. A Method for Expanding a Simple Alternant of the Third Order

The method of attack used in this paper will now be demonstrated by expanding $|5,3,0|$. On subtracting the first column from the other two columns, it is found that

$$|5,3,0| = \begin{vmatrix} 1 & 1 & 1 \\ a^3 & b^3 & c^3 \\ a^5 & b^5 & c^5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a^3 & b^3-a^3 & c^3-a^3 \\ a^5 & b^5-a^5 & c^5-a^5 \end{vmatrix}.$$

The factors $(b-a)$ and $(c-a)$ can be removed, and the determinant expanded according to the elements of the first row, leaving

$$|5,3,0| = D' \begin{vmatrix} a^2+ab+b^2 & a^2+ac+c^2 \\ a^4+a^3b+a^2b^2+ab^3+b^4 & a^4+a^3c+a^2c^2+ac^3+c^4 \end{vmatrix}$$

$$= D' \begin{vmatrix} a^2+ab+b^2 & a^2+ac+c^2 \\ a^2(a^2+ab+b^2)+ab^3+b^4 & a^2(a^2+ac+c^2)+ac^3+c^4 \end{vmatrix}$$

$$= a^2D' \begin{vmatrix} a^2+ab+b^2 & a^2+ac+c^2 \\ a^2+ab+b^2 & a^2+ac+c^2 \end{vmatrix} + D' \begin{vmatrix} a^2+ab+b^2 & a^2+ac+c^2 \\ ab^3+b^4 & ac^3+c^4 \end{vmatrix}$$

$$= D' \begin{vmatrix} a^2+ab+b^2 & a^2+ac+c^2 \\ ab^3+b^4 & ac^3+c^4 \end{vmatrix}.$$

So $|5,3,0| = D'[(a^2+ab+b^2)(ac^3+c^4) - (a^2+ac+c^4)(ab^3+b^4)]$. The expression inside the brackets is an alternating function in b and c. Hence, after the multiplication is performed, the terms can be paired in such a way that $(c-b)$ can be factored out. This leaves a symmetric function in a, b, and c. After the afore-mentioned operations are carried out, $|5,3,0|$ becomes

$$D[\Sigma a^3b^2 + \Sigma a^3bc + 2\Sigma a^2b^2c].$$

IV. Evaluations of Some Special Alternants

The simplest case of the simple alternant of the third order is the one which has the general form of $|n,1,0|$. Some examples with their expansions are

$|2,1,0| = D,$

$|3,1,0| = D(\Sigma a),$

$|4,1,0| = D(\Sigma a^2 + \Sigma ab),$

$|5,1,0| = D(\Sigma a^3 + \Sigma a^2 b + \Sigma abc),$

$|6,1,0| = D(\Sigma a^4 + \Sigma a^3 b + \Sigma a^2 b^2 + \Sigma a^2 bc),$

$|7,1,0| = D(\Sigma a^5 + \Sigma a^4 b + \Sigma a^3 b^2 + \Sigma a^3 bc + \Sigma a^2 b^2 c),$

and

$|8,1,0| = D(\Sigma a^6 + \Sigma a^5 b + \Sigma a^4 b^2 + \Sigma a^4 bc + \Sigma a^3 b^3 + \Sigma a^3 b^2 c + \Sigma a^2 b^2 c^2).$

An expansion of this general form is

$$|n,1,0| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^n & b^n & c^n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^n & b^n-a^n & c^n-a^n \end{vmatrix}$$

$$= D' \begin{vmatrix} 1 & 1 \\ a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1} & a^{n-1}+a^{n-2}c+a^{n-3}c^2+\ldots+c^{n-1} \end{vmatrix}$$

$$= D'[a^{n-2}(c-b)+a^{n-3}(c^2-b^2)+a^{n-4}(c^3-b^3)+\ldots+(c^{n-1}-b^{n-1})]$$

$$= D[a^{n-2}+a^{n-3}(b+c)+a^{n-4}(b^2+bc+c^2)+\ldots$$
$$+(b^{n-2}+b^{n-3}c+b^{n-4}c^2+\ldots+c^{n-2})].$$

$$|n,1,0| = D[\Sigma a^{n-2} + \Sigma a^{n-3}b + \Sigma a^{n-4}b^2 + \Sigma a^{n-4}bc + \ldots]$$

$$= D[\textstyle\sum \Sigma a^i b^j c^k],$$

where i, j, and k take on all values that satisfy the conditions $n-2 \geqq i \geqq j \geqq k$ and $i + j + k = n - 2$.

The second simplest case is the one which has the form $|n,2,0|$. Some examples with their expansions are

$|3,2,0| = D(\Sigma ab)$,

$|4,2,0| = D(\Sigma a^2b + 2\Sigma abc)$,

$|5,2,0| = D(\Sigma a^3b + \Sigma a^2b^2 + 2\Sigma a^2bc)$,

$|6,2,0| = D(\Sigma a^4b + \Sigma a^3b^2 + 2\Sigma a^3bc + 2\Sigma a^2b^2c)$,

$|7,2,0| = D(\Sigma a^5b + \Sigma a^4b^2 + 2\Sigma a^4bc + \Sigma a^3b^3 + 2\Sigma a^3b^2c + 2\Sigma a^2b^2c^2)$,

and

$|8,2,0| = D(\Sigma a^6b + \Sigma a^5b^2 + 2\Sigma a^5bc + \Sigma a^4b^3 + 2\Sigma a^4b^2c + 2\Sigma a^3b^3c + 2\Sigma a^3b^2c^2)$.

An expansion of this general form is

$$|n,2,0| = D'\begin{vmatrix} a+b & a+c \\ a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1} & a^{n-1}+a^{n-2}c+a^{n-3}c^2+\ldots+c^{n-1} \end{vmatrix}$$

$$= D'\begin{vmatrix} a+b & a+c \\ a^{n-3}b^2+a^{n-4}b^3+a^{n-5}b^4+\ldots+b^{n-1} & a^{n-3}c^2+a^{n-4}c^3+a^{n-5}c^4+\ldots+c^{n-1} \end{vmatrix}$$

$$= D'[a^{n-2}(c^2-b^2)+a^{n-3}(c^3-b^3)+\ldots+a^2(c^{n-2}-b^{n-2})+a(c^{n-1}-b^{n-1})$$
$$+a^{n-3}bc(c-b)+a^{n-4}bc(c^2-b^2)+\ldots+abc(c^{n-3}-b^{n-3})$$
$$+bc(c^{n-2}-b^{n-2})]$$

$$= D[a^{n-2}(b+c)+a^{n-3}(b^2+bc+c^2)+\ldots+a^2(b^{n-3}+b^{n-4}c+\ldots+c^{n-3})$$
$$+a(b^{n-2}+b^{n-3}c+\ldots+c^{n-2})+a^{n-3}bc+a^{n-4}bc(b+c)+\ldots$$
$$+abc(b^{n-4}+b^{n-5}c+\ldots+c^{n-4})+bc(b^{n-3}+b^{n-4}c+\ldots+c^{n-3})]$$

$$= D[\Sigma a^{n-2}b + \Sigma a^{n-3}b^2 + \Sigma a^{n-4}b^3 + \ldots + 2(\Sigma a^{n-3}bc + \Sigma a^{n-4}b^2c$$
$$+ \Sigma a^{n-5}b^3c + \ldots + \Sigma a^{n-5}b^2c^2 + \Sigma a^{n-6}b^3c^2 + \Sigma a^{n-7}b^4c^2 + \ldots$$
$$+ \Sigma a^{n-7}b^3c^3 + \Sigma a^{n-8}b^4c^3 + \Sigma a^{n-9}b^5c^3 + \ldots)].$$

$$|n,2,0| = D(\sum \Sigma a^ib^j + \sum 2\Sigma a^rb^sc^t),$$

where $n-2 \geqq i \geqq j$ and $n-3 \geqq r \geqq s \geqq t$, with $i+j = r+s+t = n-1$.

The third simplest case will now be the alternants of the form $|n,3,0|$. Some examples with their evaluations are

$|4,3,0| = D(\Sigma a^2b^2 + \Sigma a^2bc)$,

$|5,3,0| = D(\Sigma a^3b^2 + \Sigma a^3bc + 2\Sigma a^2b^2c)$,

$|6,3,0| = D(\Sigma a^4b^2 + \Sigma a^4bc + \Sigma a^3b^3 + 2\Sigma a^3b^2c + 3\Sigma a^2b^2c^2)$,

$|7,3,0| = D(\Sigma a^5b^2 + \Sigma a^5bc + \Sigma a^4b^3 + 2\Sigma a^4b^2c + 2\Sigma a^3b^3c + 3\Sigma a^3b^2c^2)$,

$|8,3,0| = D(\Sigma a^6b^2 + \Sigma a^6bc + \Sigma a^5b^3 + 2\Sigma a^5b^2c + \Sigma a^4b^4 + 2\Sigma a^4b^3c + 3\Sigma a^4b^2c^2 + 3\Sigma a^3b^3c^2)$,

and

$|9,3,0| = D(\Sigma a^7b^2 + \Sigma a^7bc + \Sigma a^6b^3 + 2\Sigma a^6b^2c + \Sigma a^5b^4 + 2\Sigma a^5b^3c + 3\Sigma a^5b^2c^2 + 2\Sigma a^4b^4c + 3\Sigma a^4b^3c^2 + 3\Sigma a^3b^3c^3)$.

An evaluation of the general form is

$$|n,3,0| = D' \begin{vmatrix} a^2+ab+b^2 & a^2+ac+c^2 \\ a^{n-1}+a^{n-2}b+a^{n-3}b^2+a^{n-4}b^3+\dots+b^{n-1} & a^{n-1}+a^{n-2}c+a^{n-3}c^2+a^{n-4}c^3+\dots+c^{n-1} \end{vmatrix}$$

$$= D'[a^{n-2}(c^3-b^3) + a^{n-3}(c^4-b^4) + a^{n-4}(c^5-b^5) + \dots + a^2(c^{n-1}-b^{n-1}) + a^{n-3}bc(c^2-b^2) + a^{n-4}bc(c^3-b^3) + a^{n-5}bc(c^4-b^4) + \dots + abc(c^{n-2}-b^{n-2}) + a^{n-4}b^2c^2(c-b) + a^{n-5}b^2c^2(c^2-b^2) + a^{n-6}b^2c^2(c^3-b^3) + \dots + b^2c^2(c^{n-3}-b^{n-3})]$$

$$= D\{a^{n-2}(b^2+bc+c^2) + a^{n-3}[(b^3+b^2c+bc^2+c^3) + bc(b+c)] + a^{n-4}[(b^4+b^3c+b^2c^2+bc^3+c^4) + bc(b^2+bc+c^2) + b^2c^2] + a^{n-5}[(b^5+b^4c+\dots+c^5) + bc(b^3+b^2c+bc^2+c^3) + b^2c^2(b+c)] + \dots + a^2[(b^{n-2}+b^{n-3}c+\dots+c^{n-2}) + bc(b^{n-4}+b^{n-5}c+\dots+c^{n-4}) + b^2c^2(b^{n-6}+b^{n-7}c+\dots+c^{n-6})] + a[bc(b^{n-3}+b^{n-4}c+\dots+c^{n-3}) + b^2c^2(b^{n-5}+b^{n-6}c+\dots+c^{n-5})] + [b^2c^2(b^{n-4}+b^{n-5}c+\dots+c^{n-4})]\}.$$

$$|n,3,0| = D[\sum \Sigma a^ib^j + \sum 2\Sigma a^rb^sc + \sum 3\Sigma a^tb^uc^v],$$

where $n-2 \leq i \leq j$, $n-3 \leq r \leq s$, $n-4 \leq t \leq u \leq v$, and $i + j = r + s + 1 = t + u + v = n$.

V. An Evaluation of the General Case

An evaluation of the general simple alternant of the third order follows

$$|n,m,0| = \begin{vmatrix} 1 & 0 & 0 \\ a^m & b^m-a^m & c^m-a^m \\ a^n & b^n-a^n & c^n-a^n \end{vmatrix} = D' \begin{vmatrix} a^{m-1}+a^{m-2}b+\ldots+b^{m-1} & a^{m-1}+a^{m-2}c+\ldots+c^{m-1} \\ a^{n-1}+a^{n-2}b+\ldots+a^{n-m}b^{m-1}+a^{n-m-1}b^m+\ldots+b^{n-1} & a^{n-1}+a^{n-2}c+\ldots+a^{n-m}c^{m-1}+a^{n-m-1}c^m+\ldots+c^{n-1} \end{vmatrix}$$

$$= D' \begin{vmatrix} a^{m-1}+a^{m-2}b+\ldots+b^{m-1} & a^{m-1}+a^{m-2}c+\ldots+c^{m-1} \\ a^{n-m-1}b^m+a^{n-m-2}b^{m+1}+\ldots+b^{n-1} & a^{n-m-1}c^m+a^{n-m-2}c^{m+1}+\ldots+c^{n-1} \end{vmatrix}.$$

$$|n,m,0| = D'[(a^{m-1}+a^{m-2}b+\ldots+b^{m-1})(a^{n-m-1}c^m+a^{n-m-2}c^{m+1}+\ldots+c^{n-1}) - (a^{m-1}+a^{m-2}c+\ldots+c^{m-1})(a^{n-m-1}b^m+a^{n-m-2}b^{m+1}+\ldots+b^{n-1})]$$

$$= D'[a^{n-2}(c^m-b^m) + a^{n-3}(c^{m+1}-b^{m+1}) + a^{n-4}(c^{m+2}-b^{m+2}) + a^{n-5}(c^{m+3}-b^{m+3})+\ldots+a^m(c^{n-2}-b^{n-2}) + a^{m-1}(c^{n-1}-b^{n-1})$$

$$+ a^{n-3}bc(c^{m-1}-b^{m-1}) + a^{n-4}bc(c^m-b^m) + a^{n-5}bc(c^{m+1}-b^{m+1}) + a^{n-6}bc(c^{m+2}-b^{m+2}) + \ldots + a^{m-1}bc(c^{n-3}-b^{n-3}) + a^{m-2}bc(c^{n-2}-b^{n-2})$$

$$+ a^{n-4}b^2c^2(c^{m-2}-b^{m-2}) + a^{n-5}b^2c^2(c^{m-1}-b^{m-1}) + a^{n-6}b^2c^2(c^m-b^m) + a^{n-7}b^2c^2(c^{m+1}-b^{m+1}) + \ldots + a^{m-2}b^2c^2(c^{n-4}-b^{n-4}) + a^{m-3}b^2c^2(c^{n-3}-b^{n-3})$$

$$+ \ldots + ab^{m-2}c^{m-2}(c^{n-m+1}-b^{n-m+1}) + b^{m-2}c^{m-2}(c^{n-m+2}-b^{n-m+2})$$

$$+ b^{m-1}c^{m-1}(c^{n-m}-b^{n-m})].$$

$$|n,m,0| = D\{[a^{n-2}(b^{m-1}+b^{m-2}c+b^{m-3}c^2+\ldots+bc^{m-2}+c^{m-1})]$$

$$+ a^{n-3}[(b^m+b^{m-1}c+b^{m-2}c^2+\ldots+bc^{m-1}+c^m) + bc(b^{m-2}+b^{m-3}c+b^{m-4}c^2+\ldots+bc^{m-3}+c^{m-2})]$$

$$+ a^{n-4}[(b^{m+1}+b^mc+b^{m-1}c^2+\ldots+c^{m+1}) + bc(b^{m-1}+b^{m-2}c+b^{m-3}c^2+\ldots+c^{m-1}) + b^2c^2(b^{m-3}+b^{m-4}c+b^{m-5}c^2+\ldots+c^{m-3})]$$

$$+ a^{n-5}[(b^{m+2}+b^{m+1}c+b^mc^2+\ldots+c^{m+2}) + bc(b^m+b^{m-1}c+b^{m-2}c^2+\ldots+c^m) + b^2c^2(b^{m-2}+b^{m-3}c+\ldots+c^{m-2}) + b^3c^3(b^{m-4}+b^{m-5}c+\ldots+c^{m-4})$$

$$+ a^{n-6}[(b^{m+3}+b^{m+2}c+\ldots+c^{m+3}) + bc(b^{m+1}+b^mc+\ldots+c^{m+1}) + b^2c^2(b^{m-1}+b^{m-2}c+\ldots+c^{m-1}) + b^3c^3(b^{m-3}+b^{m-4}c+\ldots+c^{m-3}) + b^4c^4(b^{m-5}+b^{m-6}c+\ldots+c^{m-5})]$$

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$$+ a[b^{m-2}c^{m-2}(b^{n-m}+b^{n-m-1}c+b^{n-m-2}c^2\ldots+c^{n-m}) + b^{m-1}c^{m-1}(b^{n-m-2}+b^{n-m-3}c+b^{n-m-4}c^2+\ldots+c^{n-m-2})]$$

$$+ b^{m-1}c^{m-1}(b^{n-m-1}+b^{n-m-2}c+b^{n-m-3}c^2+\ldots+c^{n-m-1})\}.$$

$$|n,m,0| = D[\Sigma a^{n-2}b^{m-1} + \Sigma a^{n-2}b^{m-2}c + \Sigma a^{n-2}b^{m-3}c^2 + \Sigma a^{n-2}b^{m-4}c^3 + \Sigma a^{n-2}b^{m-5}c^4+\ldots$$

$$+ \Sigma a^{n-3}b^m + 2\Sigma a^{n-3}b^{m-1}c + 2\Sigma a^{n-3}b^{m-2}c^2 + 2\Sigma a^{n-3}b^{m-3}c^3 + 2\Sigma a^{n-3}b^{m-4}c^4+\ldots$$

$$+ \sum a^{n-4}b^{m+1} + 2\sum a^{n-4}b^{m}c + 3\sum a^{n-4}b^{m-1}c^{2} + 3\sum a^{n-4}b^{m-2}c^{3} + 3\sum a^{n-4}b^{m-3}c^{4}$$

$$+ \ldots$$

$$+ \sum a^{n-5}b^{m+2} + 2\sum a^{n-5}b^{m+1}c + 3\sum a^{n-5}b^{m}c^{2} + 4\sum a^{n-5}b^{m-1}c^{3} + 4\sum a^{n-5}b^{m-2}c^{4}$$

$$+ \ldots$$

$$+ \sum a^{n-6}b^{m+3} + 2\sum a^{n-6}b^{m+2}c + 3\sum a^{n-6}b^{m+1}c^{2} + 4\sum a^{n-6}b^{m}c^{3} + 5\sum a^{n-6}b^{m-1}c^{4}$$

$$+ \ldots\ldots \quad].$$

The foregoing evaluation holds provided that the coefficient of each term is not allowed to exceed the smaller of the values, m and n-m. This can be indicated by

$$|n,m,0| = D[\sum (n-i-1) \sum a^{i}b^{n+m-(i+j+3)}c^{j}],$$

where i and j take on all positive integral values which meet the condition that

$$n-2 \geqq i \geqq n+m-(i+j+3) \geqq j,$$

and the limiting value of (n-i-1) is the least of n-m, m, and j+1.

VI. Corollary I

A corollary to the afore-mentioned result is the relationship between the conjugate simple alternants of the third order, $|n,m,0|$ and $|n,n-m,0|$.

$$|n,n-m,0| = \begin{vmatrix} 1 & 1 & 1 \\ a^{n-m} & b^{n-m} & c^{n-m} \\ a^n & b^n & c^n \end{vmatrix}$$

$$= (abc)^n \begin{vmatrix} a^{-n} & b^{-n} & c^{-n} \\ a^{-m} & b^{-m} & c^{-m} \\ 1 & 1 & 1 \end{vmatrix}.$$

After interchanging the first row with the third row and the first column with the third column,

$$|n,n-m,0| = (abc)^n \begin{vmatrix} 1 & 1 & 1 \\ c^{-m} & b^{-m} & a^{-m} \\ c^{-n} & b^{-n} & a^{-n} \end{vmatrix}.$$

Now let $x = c^{-1}$, $y = b^{-1}$, and $z = a^{-1}$ so that

$$|n,n-m,0| = (xyz)^{-n} \begin{vmatrix} 1 & 1 & 1 \\ x^m & y^m & z^m \\ x^n & y^n & z^n \end{vmatrix}$$

$$= (xyz)^{-n}(z-x)(z-y)(y-x)F(x,y\ z),$$

where $F(x,y\ z)$ is a symmetric function.

After re-introducing the original variables

$$|n,n-m,0| = (abc)^n(a^{-1}-c^{-1})(a^{-1}-b^{-1})(b^{-1}-c^{-1})F(\tfrac{1}{a},\tfrac{1}{b},\tfrac{1}{c})$$

$$= (abc)^{n-2}(c-a)(b-a)(c-b)F(\tfrac{1}{a},\tfrac{1}{b},\tfrac{1}{c}),$$

where the symmetric function is the one obtained in the evaluation of $|n,m,0|$.

VII. Corollary II

Another corollary is a consideration of the simple alternant of the third order in which each element is a single variable raised to a negative integral power. The general case can be put into the form $|-n, m-n, 0|$, where n and m are positive integers and $n > m$. Then

$$|-n, m-n, 0| = \begin{vmatrix} 1 & 1 & 1 \\ a^{m-n} & b^{m-n} & c^{m-n} \\ a^{-n} & b^{-n} & c^{-n} \end{vmatrix}$$

$$= (abc)^{-n} \begin{vmatrix} a^{n} & b^{n} & c^{n} \\ a^{m} & b^{m} & c^{m} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -(abc)^{-n} \begin{vmatrix} 1 & 1 & 1 \\ a^{m} & b^{m} & c^{m} \\ a^{n} & b^{n} & c^{n} \end{vmatrix}.$$

This makes $|-n, m-n, 0| = -(abc)^{-n}|n, m, 0|$.

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